Available online at www.sciencedirect.com

ScienceDirect

ELSEVIER

European Journal of Combinatorics 29 (2008) 242–253

European Journal of Combinatorics

www.elsevier.com/locate/ejc

A coordinatization structure for generalized quadrangles with a regular spread

Bart De Bruyn[1]

Department of Pure Mathematics and Computer Algebra, Ghent University, Galglaan 2, B-9000 Ghent, Belgium

Received 13 July 2005; accepted 25 September 2006
Available online 7 November 2006

Abstract

We shall coordinatize generalized quadrangles with a regular spread by means of a Steiner system (P, L), a set X and a certain nice map $\Delta : P \times P \to \mathrm{Sym}(X)$. We shall then show how this coordinatization method can be used to improve a result independently obtained by Kantor [W.M. Kantor, Note on span-symmetrical generalized quadrangles, Adv. Geom. 2 (2) (2002) 197–200] and Thas [K. Thas, Classification of span-symmetric generalized quadrangles of order s, Adv. Geom. 2 (2) (2002) 189–196] stating that a generalized quadrangle of order $s \geq 2$ is isomorphic to $W(s)$ if it has a hyperbolic line all of whose points are centres of symmetry. We shall show that if a generalized quadrangle Q of order $s \geq 2$ has a hyperbolic line containing only regular points, then all these points are also centres of symmetry. Combining this with the above-mentioned result independently obtained by Kantor and Thas, we then obtain that Q is isomorphic to the symplectic quadrangle $W(s)$.

1. Introduction

A *generalized quadrangle* of order (s, t), $s, t \in \mathbb{N} \setminus \{0\}$, or shortly a GQ$(s, t)$, is a partial linear space Q which satisfies the following properties: (a) each line is incident with precisely $s + 1$ points; (b) each point is incident with precisely $t + 1$ lines; (c) for every line L and every point p not incident with L, there exists a unique line through p meeting L. If $s = t$, then we also say that Q has order s. A *triad* of a generalized quadrangle Q is a set $T = \{x, y, z\}$ of three mutually noncollinear points. Any point of Q collinear with x, y and z is called a *center* of T.

E-mail address: bdb@cage.ugent.be.
[1] Postdoctoral Researcher of the Research Foundation — Flanders.

0195-6698/$ - see front matter
doi:10.1016/j.ejc.2006.09.004

If K and L are two nonconcurrent lines of a generalized quadrangle and if x is a point of K, then $p_{K \to L}(x)$ denotes the unique point of L collinear with x. We define $p_{K \to K}(x) = x$ for every line K and every point x of K.

Let Q be a generalized quadrangle of order (s, t). For every point x of Q, let $x^\perp$ denote the set of all points collinear with x (so $x \in x^\perp$). If X is a nonempty set of vertices of Q, then we define $X^\perp := \bigcap_{x \in X} x^\perp$ and $X^{\perp\perp} := (X^\perp)^\perp$ if $X^\perp \neq \emptyset$. If x and y are two different points, then $|\{x, y\}^\perp|$ is equal to either $s + 1$ or $t + 1$ depending on whether x and y are collinear or not. The set $\{x, y\}^{\perp\perp}$ is called the *span* of the pair (x, y). If x and y are collinear, then $\{x, y\}^{\perp\perp}$ coincides with the line xy. If x and y are not collinear, then $\{x, y\}^{\perp\perp}$ is also called the *hyperbolic line* through x and y; since $\{x, y\}^\perp$ contains two noncollinear points, this hyperbolic line contains at most $t + 1$ points. If the hyperbolic line through two noncollinear points x and y contains precisely $t + 1$ points, then the pair (x, y) is called *regular*. A point x is called *regular* if the pair (x, y) is regular for every point y not collinear with x. If x is a regular point of a generalized quadrangle of order $s \geq 2$, then the incidence structure π_x with points the elements of $x^\perp$ and with lines all the spans $\{a, b\}^{\perp\perp}$, $a, b \in x^\perp$ with $a \neq b$, is a projective plane of order s, see [6, 1.3.1]. The point-line dual of a GQ(s, t) is a GQ(t, s); so, the notion of regularity can also be defined for the lines of a generalized quadrangle.

A *spread* S of a generalized quadrangle Q is a set of lines partitioning the point set of Q. A spread S is called *regular* if the following holds for any two different lines K and L of S: (i) (K, L) is regular, (ii) $\{K, L\}^{\perp\perp} \subseteq S$. A spread S of Q is called a *spread of symmetry* if for every line $K \in S$ and all $k_1, k_2 \in K$, there exists an automorphism of Q fixing each line of S and mapping k_1 to k_2. Every spread of symmetry is a regular spread. If S is a spread of a GQ(s, t) with $t \neq 1$, then by Theorem 4.1 of [1], there exist at most $s+1$ automorphisms of the GQ which fix each line of S, and equality holds if and only if S is a *spread of symmetry*.

If x is a point of a GQ(s, t) with $s \neq 1$, then there are at most t automorphisms of the GQ which fix every point of $x^\perp$, see [6]. If there are precisely t such automorphisms, then x is called a *centre of symmetry*. A centre of symmetry is always a regular point.

If x is a regular point of a generalized quadrangle Q of order s with $s \neq 1$, then a new generalized quadrangle $P(Q, x)$ can be derived from it, see [5] or [6, 3.1.4]. The points of $P(Q, x)$ are the points of Q not collinear with x and the lines of $P(Q, x)$ are, on the one hand, the lines of Q not containing x and on the other hand the hyperbolic lines of Q through x (natural incidence). $P(Q, x)$ is a generalized quadrangle of order $(s - 1, s + 1)$. The hyperbolic lines through x define a regular spread $S(Q, x)$ of $P(Q, x)$. By Theorems 2.7 and 2.8 of [3], the following are equivalent: (i) x is a centre of symmetry, (ii) $S(Q, x)$ is a spread of symmetry of $P(Q, x)$.

Let Q be a generalized quadrangle of order $s \geq 2$ with a regular point x and let y be a point of Q noncollinear with x. Let u, v and w denote three points of Q noncollinear with x such that $y \sim u \sim v \sim w \sim y$, $y \not\sim v$ and $u \not\sim w$. By [6, 1.3.6], the triad $\{x, y, v\}$ has either 1 or $s + 1$ centres. Now, the points u and w are collinear with y and v, but not with x. So, the triad $\{x, y, v\}$ has a unique centre a_1. In a similar way, one shows that the triad $\{x, u, w\}$ has a unique centre a_2. Obviously, $a_1 \neq a_2$. If the points x, a_1 and a_2 are collinear for all possible choices for u, v and w such that $y \sim u \sim v \sim w \sim y$, $x \not\sim u$, $x \not\sim v$, $x \not\sim w$, $y \not\sim v$ and $u \not\sim w$, then we say that the pair (Q, x) satisfies property (P_y).

Theorem 1 *([2]). Let Q be a generalized quadrangle of order $s \geq 2$ with a regular point x and let y be a point of Q noncollinear with x. Then the pair (Q, x) satisfies property (P_y) if and only if y is a regular point.*

The generalized quadrangle $W(q)$, q prime power, is the GQ of the totally isotropic points and lines of a symplectic polarity of $PG(3, q)$. Every point of $W(q)$ is regular (even a center of symmetry). Kantor and Thas independently proved the following.

Theorem 2 (*[4,7]*). *Let Q denote a generalized quadrangle of order $s \geq 2$. If Q has a hyperbolic line all of whose points are centres of symmetry, then s is a prime power and Q is isomorphic to $W(s)$.*

In the present paper we will improve this result as follows:

Main Theorem. *Let Q be a generalized quadrangle of order $s \geq 2$ having a hyperbolic line containing only regular points, then each of these regular points is a center of symmetry. As a consequence, Q is isomorphic to $W(s)$.*

In Section 8, we will briefly discuss the case in which there exists a hyperbolic line $\mathcal{H}$ with fewer than $s + 1$ regular points. We will show there that s regular points in $\mathcal{H}$ are already sufficient to conclude that Q is isomorphic to $W(s)$. Further improvements might still be possible.

Our (Improved) Main Theorem characterizes $W(s)$ in terms of a nice set of roughly s regular points. The other known characterizations of $W(s)$ in terms of a nice set of regular points are collected in Theorem 3. Notice that each of these characterizations needs roughly s^2 regular points. Notice also that our Main Theorem is an improvement of (a), but not of (b) and (c).

Theorem 3 (*[6, 1.3.6, 5.2.5 and 5.2.6]*). *Let Q denote a generalized quadrangle of order $s \geq 2$.*

(a) *If there exists a point x such that every point of $x^\perp \setminus \{x\}$ is regular, then s is a prime power and Q is isomorphic to $W(s)$.*
(b) *If there exists an ovoid O, each point of which is regular, then s is an even prime power and Q is isomorphic to $W(s)$.*
(c) *If there exists a subquadrangle of order $(s, 1)$, each point of which is regular, then s is an even prime power and Q is isomorphic to $W(s)$.*

The proof of the characterization of $W(s)$ in terms of a hyperbolic line $\mathcal{H}$ with only regular points consists of the following two steps.

(1) Every point of $\mathcal{H}$ is a centre of symmetry.
(2) A generalized quadrangle of order $s \geq 2$ is isomorphic to $W(s)$ if it has a hyperbolic line containing only centres of symmetries.

The second step has been realized by Kantor [4] and Thas [7] with the aid of group theory. It is interesting to observe that one can give a short and very elementary proof of the second step in the case that one of the projective planes π_x, $x \in \mathcal{H}$, is Desarguesian, see [2]. In order to realize the first step, we will introduce a coordinatization method for generalized quadrangles with a regular spread. We will coordinatize these quadrangles by certain objects which we will call ‘generalized admissible triples’. These objects generalize the so-called ‘admissible triples’ which were introduced in [1] in order to coordinatize generalized quadrangles with a spread of symmetry. In Section 6 we will characterize admissible triples among all generalized admissible triples and we will use these characterizations in Section 7 to prove our Main Theorem.

2. Generalized admissible triples

Following [1], an *admissible triple* is a triple $T = (\mathcal{L}, G, \Delta)$, where

(i) G is a nontrivial group. We put $s := |G| - 1$. We will use the multiplicative notation for G.
(ii) $\mathcal{L}$ is a Steiner system $S(2, s + 1, st + 1)$, i.e., a linear space of order $(s, t - 1)$ $(s, t \geq 1)$. We denote the point set of $\mathcal{L}$ by P.
(iii) Δ is a map from $P \times P$ to G such that the following holds for any three points x, y and z of $\mathcal{L}$:

$$x, y \text{ and } z \text{ are collinear } \Leftrightarrow \Delta(x, y)\Delta(y, z) = \Delta(x, z).$$

Admissible triples were introduced in [1] and used there to coordinatize generalized quadrangles with a spread of symmetry. We will now generalize the notion of admissible triple in order to coordinatize generalized quadrangles with a regular spread. This generalization forces us however to adopt a slightly different but equivalent definition for the notion of admissible triple.

A *generalized admissible triple* is a triple $T = (\mathcal{L}, X, \Delta)$, where

- X is a set of size at least 2. We put $s := |X| - 1$. Let $\mathrm{Sym}(X)$ denote the group of permutations of X. If $\sigma_1, \sigma_2 \in \mathrm{Sym}(X)$, then we denote $\sigma_2 \circ \sigma_1$ also by $\sigma_1\sigma_2$. We denote the trivial automorphism of X by 1.
- $\mathcal{L}$ is a Steiner system $S(2, s + 1, st + 1)$, $s, t \geq 1$. We denote the point set of $\mathcal{L}$ by P.
- Δ is a map from $P \times P$ to $\mathrm{Sym}(X)$ such that the following conditions hold for any three points p, q and r of $\mathcal{L}$:
  (GAT1) If p, q and r are collinear, then $\Delta(p, q)\Delta(q, r) = \Delta(p, r)$.
  (GAT2) If p, q and r are not collinear, then the permutation $\Delta(p, q)\Delta(q, r)\Delta(r, p)$ has no fixpoints.

Suppose that $T = (\mathcal{L}, X, \Delta)$ is a generalized admissible triple.

- By taking $p = q$ in (GAT1), we see that $\Delta(p, p) = 1$ for every point p of $\mathcal{L}$.
- By taking $p = r$ in (GAT1), we then see that $\Delta(q, p) = \Delta(p, q)^{-1}$ for every two points p and q of $\mathcal{L}$.

A generalized admissible triple $T = (\mathcal{L}, X, \Delta)$ is called an *admissible triple* if there exists a binary operation $\cdot$ on the set X such that:

(AT1) $(X, \cdot)$ is a (multiplicative) group,
(AT2) $x^{-1} \cdot x^{\Delta(p_1,p_2)}$ only depends on the points p_1 and p_2 of $\mathcal{L}$ and not on the element x of X.

Putting $x_{p_1p_2} := x^{-1} \cdot x^{\Delta(p_1,p_2)}$, we see that $x^{\Delta(p_1,p_2)} = x \cdot x_{p_1p_2}$ for every $x \in X$ and all points p_1 and p_2 of $\mathcal{L}$. Conditions (GAT1) and (GAT2) then become: three points p, q and r of $\mathcal{L}$ are collinear if and only if $x_{pq} \cdot x_{qr} = x_{pr}$. This indeed shows that the new definition of admissible triple is equivalent with the one given above. In the sequel, we will always use the new definition.

3. Generalized admissible triples and GQ's with a regular spread

Theorem 4. *Suppose that $T = (\mathcal{L}, X, \Delta)$ is a generalized admissible triple and let P denote the point set of $\mathcal{L}$. Let Γ be the graph with vertex set $X \times P$, with two vertices (x_1, p_1) and (x_2, p_2) adjacent whenever either:*

(a) *$p_1 = p_2$ and $x_1 \neq x_2$, or*

(b) $p_1 \neq p_2$ *and* $x_2 = x_1^{\Delta(p_1,p_2)}$.

Then Γ is the collinearity graph of a generalized quadrangle Q_T of order (s,t). Moreover, the set $L_p := \{(x,p) \mid x \in X\}$ is a line of Q_T for every point p of $\mathcal{L}$ and the lines L_p, $p \in P$, form a regular spread S_T of Q_T.

Proof. Every pair of adjacent vertices of Γ has one of the above types (a) or (b). It is easily seen that if A is a set of three mutually adjacent vertices of Γ, then every element of $\binom{A}{2}$ has the same type. So, a maximal clique in Γ is either of type (a) or (b). Obviously, the lines L_p, $p \in P$, are the only maximal cliques of type (a). Suppose now that (x_1, p_1), (x_2, p_2) and (x_3, p_3) are three mutually adjacent vertices of type (b). Then $x_2 = x_1^{\Delta(p_1,p_2)}$, $x_3 = x_2^{\Delta(p_2,p_3)} = x_1^{\Delta(p_1,p_2)\Delta(p_2,p_3)}$ and $x_3 = x_1^{\Delta(p_1,p_3)}$. So, x_1 is a fixpoint of $\Delta(p_1,p_2)\Delta(p_2,p_3)\Delta(p_3,p_1)$. Hence, p_1, p_2 and p_3 are collinear. For every element $x \in X$ and every flag (p, M) in $\mathcal{L}$, we now have the following maximal clique of type (b): $L[x,p,M] := \{(x^{\Delta(p,r)}, r) \mid r \in M\}$. Conversely, every maximal clique of type (b) is of this form. Now, let Q_T be the incidence structure with points, respectively lines, the vertices, respectively maximal cliques, of Γ. Then we know the following about Q_T (s and t are as in Section 2):

- Q_T has $(s+1)(st+1)$ points,
- every line of Q_T contains $s+1$ vertices,
- every point of Q_T is contained in $t+1$ maximal cliques,
- Q_T has no triangles.

This is sufficient information to conclude that Q_T is a generalized quadrangle of order (s,t). Obviously, the lines L_p, $p \in P$, form a spread S_T of Q_T. If L_{p_1} and L_{p_2} are two different lines of S_T, then $\{L_{p_1}, L_{p_2}\}^\perp = \{L[x, p_1, p_1p_2] \mid x \in X\}$ and $\{L_{p_1}, L_{p_2}\}^{\perp\perp} = \{L_p \mid p \in p_1p_2\}$. This proves that S_T is a regular spread of Q_T. □

Definition. For every generalized admissible triple T, we define $\Omega(T) := (Q_T, S_T)$ where Q_T and S_T are as in the previous theorem.

Theorem 5. *If T is an admissible triple then S_T is a spread of symmetry of Q_T.*

Proof. This was already shown in Theorem 3.2 of [1]. The theorem is an immediate corollary of the following observation: for every $h \in X$, the bijection $(x,p) \mapsto (h \cdot x, p)$ defines an automorphism of Q_T fixing each line of S_T. Here $\cdot$ denotes a binary operation on the set X such that conditions (AT1) and (AT2) are satisfied. □

Definition. Let Q_1 and Q_2 be two generalized quadrangles and let S_i, $i \in \{1,2\}$, be a regular spread of Q_i. Then (Q_1, S_1) and (Q_2, S_2) are said to be *equivalent* if there exists an isomorphism from Q_1 to Q_2 mapping S_1 to S_2.

Theorem 6. *Let Q be a generalized quadrangle of order (s,t) with a regular spread S. Then there exists a generalized admissible triple T such that $\Omega(T)$ is equivalent with (Q,S).*

Proof. Let $\mathcal{L}$ denote the following linear space:

- the points of $\mathcal{L}$ are the $st+1$ lines of S;
- the lines of $\mathcal{L}$ are the spans $\{A,B\}^{\perp\perp}$, where A and B are two different lines of S;
- incidence is containment.

Now, choose a line K in S and let X denote the set of points of K. For every two lines L_1 and L_2 of S, we define $\Delta(L_1, L_2) = p_{K\to L_1} p_{L_1\to L_2} p_{L_2\to K} \in \mathrm{Sym}(X)$.

- Suppose that L_1, L_2 and L_3 are three lines of S contained in the same grid. Then $p_{L_1\to L_3} = p_{L_1\to L_2} p_{L_2\to L_3}$. Hence, $\Delta(L_1, L_3) = \Delta(L_1, L_2)\Delta(L_2, L_3)$.
- Suppose that $\Delta(L_1, L_2)\Delta(L_2, L_3)\Delta(L_3, L_1)$ has a fixpoint x. Let x_1 denote the unique point of L_1 nearest to x, let x_2 denote the unique point of L_2 nearest to x_1 and let x_3 denote the unique point of L_3 nearest to x_2. One sees that $p_{L_1\to L_2} p_{L_2\to L_3} p_{L_3\to L_1}$ has x_1 as fixpoint. This implies that x_1, x_2 and x_3 are collinear, i.e. L_1, L_2 and L_3 are contained in a grid.

This shows that $T = (\mathcal{L}, X, \Delta)$ is a generalized admissible triple. Let Γ denote the graph associated with this generalized admissible triple, see Theorem 4. For every point p of Q, we define $\theta(p) := (x_p, L_p)$, where x_p denotes the unique point of K nearest to p and where L_p denotes the unique line of S through p. Clearly, θ is a bijection between the point set of Q and the vertex set of Γ. Now, suppose that p and q are different collinear points of Q. If $L_p = L_q$, then obviously $\theta(p)$ and $\theta(q)$ are adjacent. If $L_p \neq L_q$, then $p \sim q$ implies that $x_p^{\Delta(L_p, L_q)} = x_q$, or equivalently, that the vertices $\theta(p)$ and $\theta(q)$ are adjacent in Γ. So, θ is an adjacency-preserving bijection between the vertex set of the point graph Γ' of Q and the vertex set of Γ. Since Γ' and Γ have the same valency $s(t + 1)$, Γ and Γ' are isomorphic. The theorem now easily follows. $\square$

4. Equivalence of generalized admissible triples

Two generalized admissible triples $T = (\mathcal{L}, X, \Delta)$ and $T' = (\mathcal{L}', X', \Delta')$ are called *isomorphic* if there exists an isomorphism α from $\mathcal{L}$ to $\mathcal{L}'$ and a bijection β between X and X' such that $\Delta'(p^\alpha, q^\alpha) = \beta^{-1}\Delta(p, q)\beta$ for all points p and q of $\mathcal{L}$. If T and T' are isomorphic, then $\Omega(T)$ is equivalent with $\Omega(T')$. We will now examine under which conditions $\Omega(T)$ and $\Omega(T')$ are equivalent. This will lead to the notion of equivalent generalized admissible triples.

The proof of the following theorem is straightforward and we leave it as an exercise to the reader. (To establish the equivalence of $\Omega(T) = (Q_T, S_T)$ and $\Omega(T') = (Q_{T'}, S_{T'})$, consider the bijection $(x, p) \mapsto (x^{\theta_p}, p^\alpha)$ between the point-sets of Q_T and $Q_{T'}$.)

Theorem 7. *Let $T = (\mathcal{L}, X, \Delta)$ be a generalized admissible triple. Let $\mathcal{L}'$ be a linear space isomorphic to $\mathcal{L}$ and let X' denote a set of the same size as X. Let α denote an isomorphism from $\mathcal{L}$ to $\mathcal{L}'$ and let θ_p be a bijection from X to X' for every point p of $\mathcal{L}$. For any two points p and q of $\mathcal{L}$, we define*

$$\Delta'(p^\alpha, q^\alpha) = \theta_p^{-1}\Delta(p, q)\theta_q.$$

Then $T' = (\mathcal{L}', X', \Delta')$ is a generalized admissible triple and $\Omega(T')$ is equivalent with $\Omega(T)$.

Definition. We say that two generalized admissible triples $T = (\mathcal{L}, X, \Delta)$ and $T' = (\mathcal{L}', X', \Delta')$ are equivalent if there exist

- an isomorphism α from $\mathcal{L}$ to $\mathcal{L}'$,
- a bijection θ_p between X and X' for every point p of $\mathcal{L}$,

such that

$$\Delta'(p^\alpha, q^\alpha) = \theta_p^{-1}\Delta(p, q)\theta_q$$

for all points p and q of $\mathcal{L}$. Isomorphism is a special type of equivalence.

Lemma 1. *Suppose $T = (\mathcal{L}, X, \Delta)$ is a generalized admissible triple and that o is a point of $\mathcal{L}$. Then there exists an admissible triple $T' = (\mathcal{L}, X, \Delta')$ equivalent with T such that $\Delta'(o, p) = 1$ for every point p of $\mathcal{L}$.*

Proof. In the previous definition, let α be the trivial automorphism of $\mathcal{L}$ and put $\theta_p := \Delta(o, p)^{-1}$ for every point p of $\mathcal{L}$. Then $\Delta'(o, p) = \Delta(o, o)\Delta(o, p)\ \Delta(o, p)^{-1} = 1$ for every point p of $\mathcal{L}$. This proves the lemma. $\square$

Theorem 8. *Let $T = (\mathcal{L}, X, \Delta)$ and $T' = (\mathcal{L}', X', \Delta')$ denote two generalized admissible triples. Then $\Omega(T)$ is equivalent with $\Omega(T')$ if and only T and T' are equivalent.*

Proof. We have shown in Theorem 7 that if T and T' are equivalent, then also $\Omega(T) = (Q_T, S_T)$ and $\Omega(T') = (Q_{T'}, S_{T'})$ are equivalent. Conversely, suppose that $\Omega(T)$ and $\Omega(T')$ are equivalent. Then there exists an isomorphism μ from Q_T to $Q_{T'}$ mapping S_T to $S_{T'}$. Since μ maps spans of lines to spans of lines, μ determines an isomorphism α from $\mathcal{L}$ to $\mathcal{L}'$ such that the line L_p of S_T is mapped to the line L_{p^α} of $S_{T'}$. Hence, for every point p of $\mathcal{L}$ there exists a bijection $\theta_p : X \to X'$ such that the point (x, p) of Q_T is mapped to the point (x^{θ_p}, p^α) of $Q_{T'}$. Now, consider two different points p_1 and p_2 of $\mathcal{L}$. Then for every $x \in X$, the points (x, p_1) and $(x^{\Delta(p_1,p_2)}, p_2)$ are collinear. Hence, also the points $(x^{\theta_{p_1}}, p_1^\alpha)$ and $(x^{\Delta(p_1,p_2)\theta_{p_2}}, p_2^\alpha)$ are collinear. So, $\theta_{p_1}\Delta'(p_1^\alpha, p_2^\alpha) = \Delta(p_1, p_2)\theta_{p_2}$. This formula remains valid if $p_1 = p_2$. As a consequence, $\Delta'(p_1^\alpha, p_2^\alpha) = \theta_{p_1}^{-1}\Delta(p_1, p_2)\theta_{p_2}$ for all points p_1 and p_2 of $\mathcal{L}$. This means that the admissible triples T and T' are equivalent. $\square$

5. Automorphisms fixing each line of a regular spread

Let $T = (\mathcal{L}, X, \Delta)$ be a generalized admissible triple and let P denote the point set of $\mathcal{L}$. For every point p of $\mathcal{L}$, let G_p denote the subgroup of $\mathrm{Sym}(X)$ generated by all permutations $\Delta(p, p_1)\Delta(p_1, p_2)\Delta(p_2, p)$, $p_1, p_2 \in P$. Let $\tilde{G}_p$ denote the group of elements of $\mathrm{Sym}(X)$ which commute with every element of G_p.

Lemma 2. *For all points p and p' of $\mathcal{L}$, $G_p \cong G_{p'}$ and $\tilde{G}_p \cong \tilde{G}_{p'}$.*

Proof. The map $\mathrm{Sym}(X) \to \mathrm{Sym}(X); \theta \mapsto \Delta(p', p)\theta\Delta(p, p')$ defines isomorphisms from G_p to $G_{p'}$ and from $\tilde{G}_p$ to $\tilde{G}_{p'}$. $\square$

Put $\Omega(T) = (Q_T, S_T)$. Let H_S denote the group of automorphisms of Q_T fixing each line of S_T.

Theorem 9. *The group H_S is isomorphic to $\tilde{G}_p$ for every point p of $\mathcal{L}$.*

Proof. Let L_p denote the line of S_T corresponding with the point p of $\mathcal{L}$. Every automorphism of Q_T fixing each line of S_T determines a permutation of the point set of L_p and hence also a permutation of the set X. Conversely, every permutation of the point set of L_p (i.e. of X) can be extended to at most one element of H_S. We will now determine the condition under which a permutation ϕ of X extends to an element θ of H_S. Take the point (x, q) of Q_T. The point of L_p nearest to (x, q) is the point $(x^{\Delta(q,p)}, p)$ which is mapped by ϕ to the point $(x^{\Delta(q,p)\phi}, p)$. Hence, θ maps (x, q) to the point $(x^{\Delta(q,p)\phi\Delta(p,q)}, q)$. Clearly, the map $(x, q) \mapsto (x^{\Delta(q,p)\phi\Delta(p,q)}, q)$ is a permutation of the point set of Q_T. Now, let (x, q) and (x', q') be collinear points of Q_T. If $q = q'$, then obviously the images of (x, q) and (x', q') are also collinear. So, suppose that

$q \neq q'$. Then: $x' = x^{\Delta(q,q')}$. Now, the points $(x^{\Delta(q,p)\phi\Delta(p,q)}, q)$ and $(x'^{\Delta(q',p)\phi\Delta(p,q')}, q')$ are collinear if and only if:

$$x^{\Delta(q,p)\phi\Delta(p,q)\Delta(q,q')} = x^{\Delta(q,q')\Delta(q',p)\phi\Delta(p,q')}.$$

Since this holds for all elements x of X, we have:

$$\phi[\Delta(p,q)\Delta(q,q')\Delta(q',p)] = [\Delta(p,q)\Delta(q,q')\Delta(q',p)]\phi,$$

for all points q and q' of $\mathcal{L}$. Hence, the permutation ϕ of X determines an element θ of H_S if and only if $\phi \in \tilde{G}_p$. This proves the theorem. $\square$

6. Characterization of admissible triples

In this section, $T = (\mathcal{L}, X, \Delta)$ denotes a generalized admissible triple. We denote the point set of $\mathcal{L}$ by P and we put $s := |X| - 1$.

Theorem 10. *If $\mathcal{L}$ is not a line, then $\langle \mathrm{Im}(\Delta)\rangle := \langle \Delta(x,y) \mid x, y \in P\rangle$ acts transitively on X. So, $|\langle \mathrm{Im}(\Delta)\rangle| \geq s + 1$.*

Proof. Let (p, L) be an antiflag of $\mathcal{L}$, let q denote an arbitrary point of L and let x denote an arbitrary element of X. If $\langle \mathrm{Im}(\Delta)\rangle$ does not act transitively on X, then since $|L| = s + 1$ and $|\{x^\phi \mid \phi \in \langle \mathrm{Im}(\Delta)\rangle\}| < s + 1$, there exist points $q_1, q_2 \in L$ such that $q_1 \neq q_2$ and

$$x^{\Delta(q,q_1)\Delta(q_1,p)} = x^{\Delta(q,q_2)\Delta(q_2,p)}.$$

Put $y := x^{\Delta(q,q_2)}$. Then $y^{\Delta(q_2,q_1)\Delta(q_1,p)\Delta(p,q_2)} = x^{\Delta(q,q_2)\Delta(q_2,q_1)\Delta(q_1,p)\Delta(p,q_2)} = x^{\Delta(q,q_1)\Delta(q_1,p)\Delta(p,q_2)} = x^{\Delta(q,q_2)\Delta(q_2,p)\Delta(p,q_2)} = x^{\Delta(q,q_2)} = y$. So, y is a fixpoint of $\Delta(q_2,q_1)\Delta(q_1,p)\Delta(p,q_2)$, contradicting the fact that p, q_1 and q_2 are not collinear. As a consequence, $\langle \mathrm{Im}(\Delta)\rangle$ acts transitively on X. $\square$

Corollary 1. *Suppose $\mathcal{L}$ is not a line and $|\langle \mathrm{Im}(\Delta)\rangle| = s + 1$. Then $\mathrm{Im}(\Delta)$ acts regularly on X.*

Theorem 11. *Suppose that $\mathcal{L}$ is not a line and that T is an admissible triple. Then $|\langle \mathrm{Im}(\Delta)\rangle| = s + 1$.*

Proof. Let $\cdot$ denote a binary operation on the set X such that (i) $(X, \cdot)$ is a group, and (ii) $x^{-1} \cdot x^{\Delta(p_1,p_2)}$ is independent from x for all points p_1 and p_2 of $\mathcal{L}$. Then for all points p_1 and p_2 of $\mathcal{L}$, there exists a unique element $x_{p_1p_2} \in X$ such that $x^{\Delta(p_1,p_2)} = x \cdot x_{p_1p_2}$ for all $x \in X$. So, for every element θ of $\langle \mathrm{Im}(\Delta)\rangle$ there corresponds a unique element x_θ such that $x^\theta = x \cdot x_\theta$ for all $x \in X$. Hence, $|\langle \mathrm{Im}(\Delta)\rangle| \leq s + 1$. The theorem now follows from Theorem 10. $\square$

Theorem 12. *Suppose $\mathcal{L}$ is not a line and $|\langle \mathrm{Im}(\Delta)\rangle| = s + 1$. Then T is an admissible triple.*

Proof. Choose an arbitrary element $\bar{x}$ in X. By Corollary 1, $\langle \mathrm{Im}(\Delta)\rangle$ acts regularly on X. So, we can define the following binary operation $\cdot$ on X: $\bar{x}^{\theta_1} \cdot \bar{x}^{\theta_2} := \bar{x}^{\theta_1\theta_2}, \forall \theta_1, \theta_2 \in \langle \mathrm{Im}(\Delta)\rangle$. Obviously, $(X, \cdot)$ is a group with neutral element $\bar{x}$. Now, for all $p_1, p_2 \in P$ and all $\theta \in \langle \mathrm{Im}(\Delta)\rangle$, we have $(\bar{x}^\theta)^{-1} \cdot (\bar{x}^\theta)^{\Delta(p_1,p_2)} = \bar{x}^{\theta^{-1}} \cdot \bar{x}^{\theta\Delta(p_1,p_2)} = \bar{x}^{\Delta(p_1,p_2)}$. So, $x^{-1} \cdot x^{\Delta(p_1,p_2)}$ only depends on p_1 and p_2 and not on x. This proves that T is an admissible triple. $\square$

Definition. For every point p of $\mathcal{L}$, we define $\Omega(p) := \{\Delta(p,q) \mid q \in P\}$. Then $\langle \mathrm{Im}(\Delta)\rangle = \langle \bigcup_{p \in P} \Omega(p)\rangle$.

Theorem 13. *Suppose that $\mathcal{L}$ is not a line and that o is a point of $\mathcal{L}$ such that $\Delta(o, p) = 1$ for every point p of $\mathcal{L}$. Then $|\Omega(p)| \geq s + 1$ for every point $p \neq o$ of $\mathcal{L}$. Moreover, if T is an admissible triple, then $\Omega(p) = \langle \mathrm{Im}(\Delta) \rangle$ for every point $p \neq o$ of $\mathcal{L}$.*

Proof. Let L denote a line through p different from op. If q_1 and q_2 are two different points of L, then $\Delta(q_1, q_2) \neq 1$ since o, q_1 and q_2 are not collinear. Now, since p, q_1 and q_2 are collinear, $\Delta(p, q_1) = \Delta(p, q_2)\Delta(q_2, q_1) \neq \Delta(p, q_2)$. As a consequence, $|\Omega(p)| \geq |\{\Delta(p, q) \mid q \in L\}| = s + 1$. If T is an admissible triple and if p is an arbitrary point of $\mathcal{L}$ different from o, then $|\langle \mathrm{Im}(\Delta) \rangle| = s + 1 \leq |\Omega(p)|$ and $\Omega(p) \subseteq \langle \mathrm{Im}(\Delta) \rangle$. Hence, $\Omega(p) = \langle \mathrm{Im}(\Delta) \rangle$. □

Theorem 14. *Suppose*

- *$\mathcal{L}$ is not a line,*
- *there exists a point o such that $\Delta(o, p) = 1$ for every point p of $\mathcal{L}$,*
- *there exists a subset Ω of size $s + 1$ of $\mathrm{Sym}(X)$ such that $\Omega(p) = \Omega$ for every point $p \neq o$ of $\mathcal{L}$.*

  Then $\langle \mathrm{Im}(\Delta) \rangle = \Omega$ and T is an admissible triple.

Proof. Since $\langle \mathrm{Im}(\Delta) \rangle = \langle \cup_{p \in P} \Omega(p) \rangle = \langle \Omega \rangle$, it suffices to show that Ω is a group, or since Ω is finite, to show that $\theta_1\theta_2 \in \Omega$ for all $\theta_1, \theta_2 \in \Omega$. We may suppose that $\theta_1 \neq 1 \neq \theta_2$. Consider a point $p \neq o$ in $\mathcal{L}$ and a line L through p different from op. By the proof of Theorem 13, it follows that $\Omega = \{\Delta(p, q) \mid q \in L\}$. Hence, there exists a unique point p' on L such that $\Delta(p, p') = \theta_1$. In a similar way, there exists a unique point p'' on L such that $\Delta(p', p'') = \theta_2$. Now, $\theta_1\theta_2 = \Delta(p, p')\Delta(p', p'') = \Delta(p, p'') \in \Omega$. This proves the theorem. □

7. Proof of the Main Theorem

Suppose that Q is a generalized quadrangle of order $s \geq 2$ having a hyperbolic line $\mathcal{H}$ consisting of $s + 1$ regular points. Let x be one of the points of $\mathcal{H}$ and consider the generalized quadrangle $P(Q, x)$ of order $(s - 1, s + 1)$. The hyperbolic lines through x define a regular spread $S(Q, x)$ of $P(Q, x)$. Let $\mathcal{L}'$ be the linear space defined on $S(Q, x)$ by all spans $\{A, B\}^{\perp\perp}$, $A, B \in S(Q, x)$ with $A \neq B$. The linear space $\mathcal{L}'$ has s^2 points and s points on every line. So, $\mathcal{L}'$ is an affine plane of order s. Each hyperbolic line G of Q through x is uniquely determined by the set $G^\perp$. So, the linear space $\mathcal{L}'$ is isomorphic to the affine plane $\mathcal{L}''$ whose lines are the points of $x^\perp \setminus \{x\}$ and whose points are the spans $\{a, b\}^{\perp\perp}$, $a, b \in x^\perp \setminus \{x\}$ with $b \notin xa$ (incidence is reverse containment). Let L denote the line of $S(Q, x)$ corresponding with the hyperbolic line $\mathcal{H}$. Put $\Pi := \langle p_{L \to K} p_{K \to K'} p_{K' \to L} \mid K, K' \in S(Q, x) \rangle$, then Π is a group of permutations of L. For all lines L_1, L_2, L_3 of $S(Q, x)$, let $\phi(L_1, L_2, L_3)$ denote the following element of Π: $p_{L \to L_1} p_{L_1 \to L_2} p_{L_2 \to L_3} p_{L_3 \to L}$.

Lemma 3. *If L_1, L_2 and L_3 are lines of $S(Q, x)$ such that L, L_1, L_2, L_3 are mutually different and $\phi(L_1, L_2, L_3)$ has a fixpoint, then the lines $\{L, L_2\}^{\perp\perp}$ and $\{L_1, L_3\}^{\perp\perp}$ of $\mathcal{L}'$ are parallel.*

Proof. Obviously, the lemma holds if L, L_1, L_2 and L_3 are contained in a grid. So, suppose that L, L_1, L_2 and L_3 are not contained in the same grid. Let $y \in L$ denote a fixpoint of $\phi(L_1, L_2, L_3)$, let u denote the unique point of L_1 collinear with y, let v denote the unique point of L_2 collinear with u and let w denote the unique point of L_3 collinear with v. Since y is a fixpoint of $\phi(L_1, L_2, L_3)$, $w \sim y$, and since L, L_1, L_2 and L_3 are not contained in the same grid, $y \not\sim v$ and $u \not\sim w$. Now, regard y, u, v and w as points of Q. Since $y \in L$, y is regular and

(Q, x) satisfies property (P_y) by Theorem 1. So, $a_1 \sim a_2$ where a_1 is the unique centre of the triad $\{x, y, v\}$ and where a_2 is the unique centre of the triad $\{x, u, w\}$. This condition means that the lines $\{L, L_2\}^{\perp\perp}$ and $\{L_1, L_3\}^{\perp\perp}$ of $\mathcal{L}'$ are parallel. $\square$

Now, since $S(Q, x)$ is a regular spread of $P(Q, x)$, there exists, up to equivalence, a unique generalized admissible triple $T = (\mathcal{L}, X, \Delta)$ such that $\Omega(T)$ is equivalent with $(P(Q, x), S(Q, x))$. Here X is a set of size s and $\mathcal{L}$ denotes a linear space isomorphic to the affine plane $\mathcal{L}'$, see the proof of Theorem 6. Let o denote the point of $\mathcal{L}$ corresponding with the point L of $\mathcal{L}'$ (given a certain isomorphism from $\mathcal{L}'$ to $\mathcal{L}$). Since T is a generalized admissible triple, we have the following properties:

(I) If p, q and r are three collinear points of $\mathcal{L}$, then $\Delta(p, q)\Delta(q, r) = \Delta(p, r)$.

(II) If p, q and r are points of $\mathcal{L}$ such that $\Delta(p, q)\Delta(q, r)\Delta(r, p)$ has a fixpoint, then p, q and r are collinear.

By Lemma 1 and Theorem 8, we may also suppose that

(III) $\Delta(o, p) = 1$ for every point p of $\mathcal{L}$.

Lemma 3 gives rise to the following condition:

(IV) if o, p, q and r are distinct points such that $\Delta(p, r)\Delta(r, q)$ has at least one fixpoint, then the line or is parallel with pq.

By properties (I), (II) and (III), $\Delta(p, q) = 1$ if and only if o, p and q are collinear.

Lemma 4. *Let a be an element of X.*

(a) *Let L denote a line through o and let p denote a point not contained on L. Then $X = \{a^{\Delta(p,l)} \mid l \in L\}$.*

(b) *Let L denote a line not containing o and let p denote a point of L. Then $X = \{a^{\Delta(p,l)} \mid l \in L\}$.*

(c) *Let p denote a point of $\mathcal{L}$ different from o and let b denote an arbitrary element of X. Then the points q of $\mathcal{L}$ for which $a^{\Delta(p,q)} = b$ form a line of $\mathcal{L}$ parallel to op.*

Proof. (a) If l_1 and l_2 are two different points of L, then $\Delta(l_1, l_2) = 1$. If $a^{\Delta(p,l_1)} = a^{\Delta(p,l_2)}$, then $\Delta(p, l_1)\Delta(l_1, l_2)\Delta(l_2, p)$ would have a as fixpoint, contradicting the fact that p, l_1 and l_2 are not collinear. So, $a^{\Delta(p,l_1)} \neq a^{\Delta(p,l_2)}$ for any two different points l_1 and l_2 of L. This implies that $X = \{a^{\Delta(p,l)} \mid l \in L\}$.

(b) Suppose the contrary. Then there exist different points l_1 and l_2 on L such that $a^{\Delta(p,l_1)} = a^{\Delta(p,l_2)}$. Since $\Delta(p, l_1)\Delta(l_1, l_2) = \Delta(p, l_2)$, $a^{\Delta(p,l_1)}$ is a fixpoint of $\Delta(l_1, l_2) = \Delta(l_1, l_2)\Delta(l_2, o)\Delta(o, l_1)$, contradicting the fact that l_1, l_2 and o are not collinear.

(c) If $b = a$, then the required points are the points of the line op. If $b \neq a$, then by (b), there exists a required point on each line through p different from op. Suppose q_1 and q_2 are two such points. Then $a^{\Delta(p,q_1)} = a^{\Delta(p,q_2)} = b$. Hence, $b^{\Delta(q_1,p)\Delta(p,q_2)} = b$. So, $op \parallel q_1q_2$ by condition (IV). The statement now easily follows. $\square$

Corollary 2. *Let p denote a point of $\mathcal{L}$ different from o and let L be a line of $\mathcal{L}$ parallel with op. Then $\Delta(p, l_1) = \Delta(p, l_2)$ for all points l_1, l_2 of L. As a consequence, $\Omega(p)$ contains precisely s elements.*

Proof. Let a denote an arbitrary element of X. By Lemma 4(c), the s points l satisfying $a^{\Delta(p,l)} = a^{\Delta(p,l_1)}$ are the s points on the unique line through l_1 parallel with op. Hence, $a^{\Delta(p,l_1)} = a^{\Delta(p,l_2)}$. Since a was arbitrary, $\Delta(p, l_1) = \Delta(p, l_2)$. By Theorem 13, it now follows that $|\Omega(p)| = s$. □

Lemma 5. *If p and q are two different points of $\mathcal{L}$ different from o, then $\Omega(p) = \Omega(q)$.*

Proof. First suppose that $q \notin op$. By symmetry, it suffices to prove that $\Omega(p) \subseteq \Omega(q)$. So, let θ denote an arbitrary element of $\Omega(p)$. We may suppose that $\theta \neq 1$. Let p' denote a point such that $\Delta(p, p') = \theta$, let $L_{p'}$ denote the unique line through p' parallel with op and let p'' denote the unique point of $L_{p'}$ such that op'' is parallel with pq. Then $\Delta(p'', q) = \Delta(p'', p)$. Hence, $\Delta(q, p'') = \Delta(p, p'') = \Delta(p, p') = \theta$. So, $\Omega(p) \subseteq \Omega(q)$. Hence, $\Omega(p) = \Omega(q)$ by symmetry.

Next, suppose that $q \in op$. If r is a point of $\mathcal{L}$ not contained in op, then $\Omega(p) = \Omega(r) = \Omega(q)$. □

Corollary 3.
- *T is an admissible triple,*
- *$S(Q, x)$ is a spread of symmetry of $P(Q, x)$,*
- *x is a centre of symmetry of Q.*

Proof. From Theorem 14, Corollary 2 and Lemma 5, it follows that T is an admissible triple. By Theorem 5, we then have that $S(Q, x)$ is a spread of symmetry of $P(Q, x)$. By [3], it then follows that x is a centre of symmetry of Q. □

Corollary 4. *The order s is a prime power and Q is isomorphic to $W(s)$.*

Proof. By Corollary 3, it follows that every point of $\mathcal{H}$ is a centre of symmetry. By Theorem 2, it then follows that Q is isomorphic to $W(s)$. □

8. Further improvements

In this section, we suppose that Q is a generalized quadrangle of order $s \geq 2$ having a hyperbolic line $\mathcal{H}$ containing $k \geq 1$ regular points. Let x be one of these regular points and consider the generalized quadrangle $P(Q, x)$ and its regular spread $S(Q, x)$. We know that there exists a generalized admissible triple $T = (\mathcal{L}, X, \Delta)$ such that $\Omega(T)$ is equivalent with $(P(Q, x), S(Q, x))$. As in Section 7, we may suppose that the following conditions hold:

(I) If p, q and r are three collinear points of $\mathcal{L}$, then $\Delta(p, q)\Delta(q, r) = \Delta(p, r)$.
(II) If p, q and r are points of $\mathcal{L}$ such that $\Delta(p, q)\Delta(q, r)\Delta(r, p)$ has a fixpoint, then p, q and r are collinear.
(III) $\Delta(o, p) = 1$ for every point p of $\mathcal{L}$.

Here, o denotes the point of $\mathcal{L}$ corresponding with $\mathcal{H}$. Condition (IV) does not necessarily hold any more and must be adapted. By the proof of Theorem 6, the set X can be identified with the set of points of the line of $S(Q, x)$ corresponding with $\mathcal{H}$. Let X' be the subset of X corresponding with the $k - 1$ regular points of $\mathcal{H} \setminus \{x\}$. The new condition then becomes:

(IV′) If o, p, q and r are distinct points such that $\Delta(p, r)\Delta(r, q)$ has at least one fixpoint belonging to X', then the line or is parallel with pq.

Lemma 4(c) then holds for all elements $b \in X'$. If Lemma 4(c) holds for all $b \in X$, then the reasoning after Lemma 4 remains valid and we can conclude that Q is isomorphic to $W(s)$.

Suppose now that $k = s$. Then Lemma 4(c) also holds for the remaining element of X. Hence, we can say the following.

Main Theorem (Improved Version). *Let Q be a generalized quadrangle of order $s \geq 2$ having a hyperbolic line containing at least s regular points, then s is a prime power and $Q \cong W(s)$.*

In the remainder of this section, we will suppose that $k = s - 1$.

Definition. Let $\mathcal{A}$ be an affine plane of order s. Let o' be a given point of $\mathcal{A}$. For every point $p \neq o'$ of $\mathcal{A}$ and for any two lines L_1 and L_2 of $\mathcal{A}$ such that $o'p$, L_1 and L_2 are mutually disjoint (and hence parallel), we consider the bipartite graph Γ_{p,L_1,L_2} with vertex set $L_1 \cup L_2$, two distinct vertices q_1 and q_2 being adjacent whenever $o' \in q_1q_2$ or $p \in q_1q_2$. We say that $(\mathcal{A}, o')$ satisfies property $(*)$ if the graph Γ_{p,L_1,L_2} is connected for all possible choices of p, L_1 and L_2. We say that $\mathcal{A}$ satisfies property $(*)$ if $(\mathcal{A}, o')$ satisfies property $(*)$ for every point o' of $\mathcal{A}$. We leave it as a straightforward exercise to the reader to verify that a Desarguesian affine plane satisfies property $(*)$ if and only if its order is a prime.

Now, return to the original problem and suppose that $(\mathcal{L}, o)$ satisfies property $(*)$. Then Lemma 4(c) holds for all elements $b \in X$.

For, let b_1 and b_2 denote the two elements of $X \setminus X'$ and suppose $b_1 \neq a \neq b_2$. The set of points q satisfying $a^{\Delta(p,q)} \in \{b_1, b_2\}$ is the union $L_1 \cup L_2$ of two lines L_1 and L_2 such that op, L_1 and L_2 are mutually disjoint. Now, Lemma 4(a)+(b) and the fact that Γ_{p,L_1,L_2} is connected imply that there exists an $i \in \{1, 2\}$ such that $L_i = \{q \mid a^{\Delta(p,q)} = b_1\}$ and $L_{3-i} = \{q \mid a^{\Delta(p,q)} = b_2\}$.

As before, we can now conclude that Q is isomorphic to $W(s)$.

References

[1] B. De Bruyn, Generalized quadrangles with a spread of symmetry, European J. Combin. 20 (8) (1999) 759–771.

[2] B. De Bruyn, S.E. Payne, Some notes on generalized quadrangles with a span of regular points, Contrib. Algebra Geom. (in press).

[3] M. De Soete, J.A. Thas, A coordinatization of generalized quadrangles of order $(s, s+2)$, J. Combin. Theory Ser. A 46 (1) (1988) 1–11.

[4] W.M. Kantor, Note on span-symmetric generalized quadrangles, Adv. Geom. 2 (2) (2002) 197–200.

[5] S.E. Payne, Nonisomorphic generalized quadrangles, J. Algebra 18 (1971) 201–212.

[6] S.E. Payne, J.A. Thas, Finite Generalized Quadrangles, in: Research Notes in Mathematics, vol. 110, Pitman, Boston, 1984.

[7] K. Thas, Classification of span-symmetric generalized quadrangles of order s, Adv. Geom. 2 (2) (2002) 189–196.